

# Higher Mathematics

## Revision Questions



Use this as part of your on-going revision

1. In collaboration with Key Facts and Methods booklet
2. Complete each question and check your working and answers.
3. To gain confidence with set questions before attempting past papers

Make sure you attempt all that you are able from a question before checking answers i.e. all parts; a), b), i), ii) etc. often you **can** complete other parts without part a).

## FORMULAE LIST

### Circle:

The equation  $x^2 + y^2 + 2gx + 2fy + c = 0$  represents a circle centre  $(-g, -f)$  and radius  $\sqrt{g^2 + f^2 - c}$ .

The equation  $(x - a)^2 + (y - b)^2 = r^2$  represents a circle centre  $(a, b)$  and radius  $r$ .

### Trigonometric formulae:

$$\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A\end{aligned}$$

**Scalar Product:**  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

or

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ where } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

### Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$ $\cos ax$	$a \cos ax$ $-a \sin ax$

### Table of standard integrals:

$f(x)$	$\int f(x) dx$
$\sin ax$ $\cos ax$	$-\frac{1}{a} \cos ax + C$ $\frac{1}{a} \sin ax + C$

# Contents Page

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Formula list Page 2

## **Expressions and Formula**

1.1 – Exponential and Logarithmic functions Page 4  
1.2 – Compound Angle Formula and Wave Function Page 6  
1.3 – Functions, inverses and graphs of functions Page 7  
1.4 – Vectors Page 7

## **Relationships and Calculus**

1.1 – Polynomials and Quadratics Page 8  
1.2 – Trigonometric Equations Page 9  
1.3 – Differentiation Page 9  
1.4 – Integration Page 10

## **Applications**

1.1 – The Straight line Page 11  
1.2 – The Circle Page 11  
1.3 – Recurrence Relations Page 12  
1.4 – Further Calculus Page 13

# Expressions and Formula

## 1.1 – Exponential and Logarithmic functions

1. Simplify

(a)  $\log_2 6 + \log_2 12 - \log_2 9$

(b)  $\frac{3}{4}\log_{10} 16 - \frac{1}{2}\log_{10} 4 + 2\log_{10} 5$

2. Solve for  $x > 0$

(a)  $\log_4 x + \log_4 (3x - 2) = 2$

(b)  $\log_3 (x^2 + x - 2) - \log_3 (x^2 - 4) = 1$

3. A curve has equation  $y = \log_2 (x + 4) - 3$ . Find where this curve cuts the x and y axes.

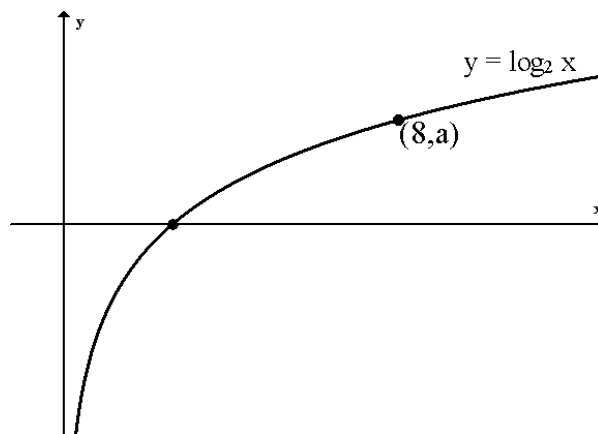
4. The diagram opposite shows the graph of  $y = \log_2 x$ .

(a) Find the value of a.

(b) Sketch the graph of  $y = \log_2 x - 1$

(c) Sketch the graph of  $y = \log_2 8x$

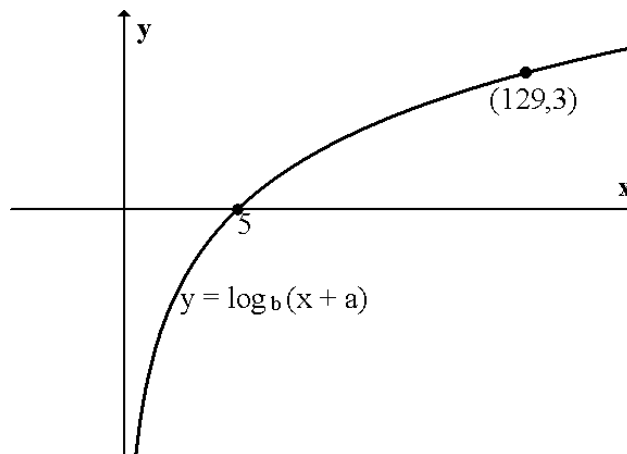
(d) Sketch the graph of  $y = \log_2 \frac{1}{x}$



5. The diagram shows the graph of  $y = \log_b(x + a)$ .

The points  $(5, 0)$  and  $(129, 3)$  lie on this graph.

Find the values of a and b.



6. The mass,  $M$  grams, of a radioactive isotope after a time of  $t$  years, is given by the formula  $M = M_0 e^{-kt}$ , where  $M_0$  is the initial mass of the isotope.

In 4 years a mass of 20 grams of the isotope is reduced to 15 grams.

(a) Calculate  $k$ .

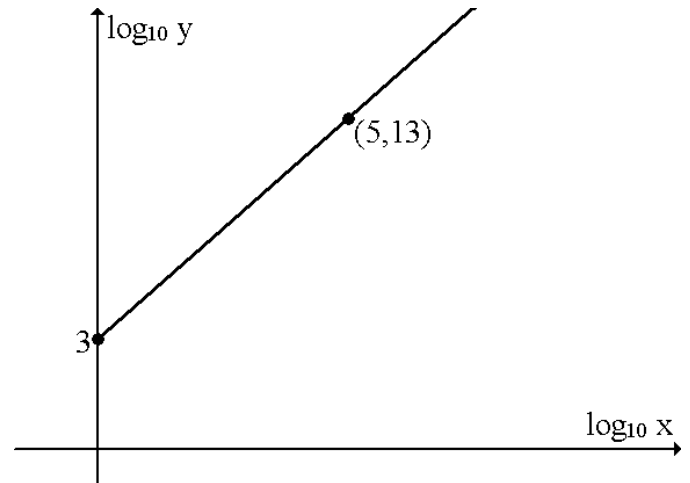
(b) Calculate the half-life of the substance i.e. the time taken for half the substance to decay.

7. Dangerous blue algae are spreading over the surface of a lake according to the formula  $A_t = A_0 e^{kt}$  where  $A_0$  is the initial area covered by the algae and  $A_t$  is the area covered after  $t$  days.  
When first noticed the algae covered an area of 100 square metres. Two weeks later the algae covered an area of 120 square metres.

- Calculate the value of  $k$ .
- The area of algae on the lake was measured on the 1<sup>st</sup> of June and again on the 1<sup>st</sup> of July. Calculate the percentage increase in area covered by the algae between these dates.

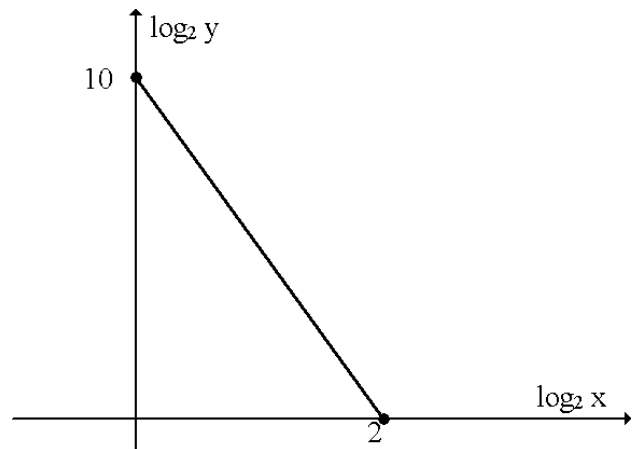
8. The graph opposite illustrates the law  $y = kx^n$ .

Find the values of  $k$  and  $n$ .



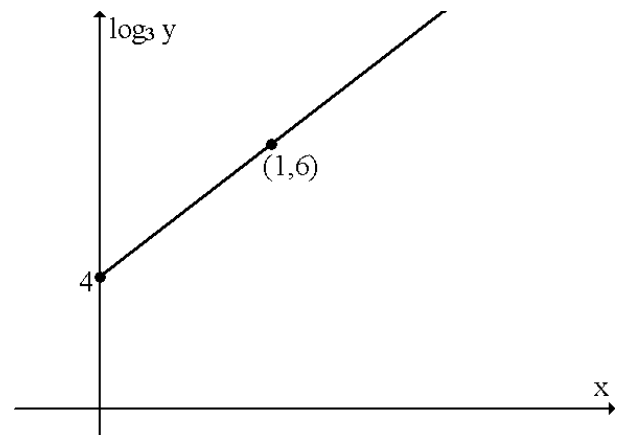
9. The graph opposite illustrates the law  $y = kx^n$ .

Find the values of  $k$  and  $n$ .



10. The graph opposite illustrates the law  $y = ab^x$ .

Find the values of  $a$  and  $b$ .



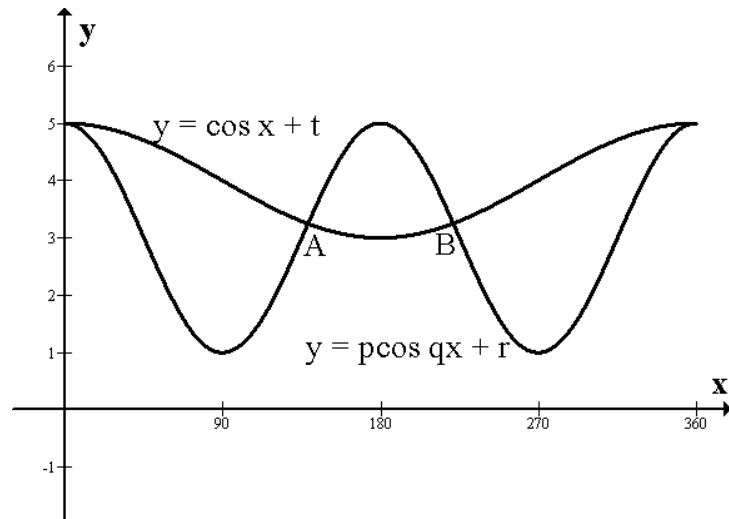
## 1.2 – Compound Angle Formula and Wave Function

1. Solve the equations

- a)  $3\tan^2 x - 1 = 0$   $0 \leq x \leq 2\pi$
- b)  $4\cos(2x - 30) + 4 = 2$   $0 \leq x \leq 360$
- c)  $3\sin 2x = 2\cos x$   $0 \leq x \leq 360$

2. The diagram opposite shows the graphs of  $y = p \cos qx + r$  and  $y = \cos x + t$ .

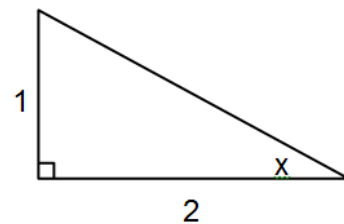
- (a) Write down the values of  $p$ ,  $q$ ,  $r$  and  $t$ .
- (b) Find the coordinates of A and B.



3. (a) Express  $4\sin x + 3\cos x$  in the form  $k\sin(x + a)$  where  $k > 0$  and  $0 \leq a \leq 360$   
(b) Solve the equation  $4\sin x + 3\cos x = 3$   $0 \leq x \leq 360$   
(c) Find the minimum value of  $4\sin x + 3\cos x$  and the value of  $x$  for which it occurs in the range  $0 \leq x \leq 360$   
(d) Sketch the graph of  $y = 4\sin x + 3\cos x$  for  $0 \leq x \leq 360$

4.  $\tan x = \frac{1}{2}$ . Find the exact value of

- (a)  $\sin 2x$
- (b)  $\cos 2x$
- (c)  $\tan 2x$



5. If  $\cos x = \frac{3}{5}$  and  $\sin y = \frac{5}{13}$ . Find the exact value of  $\cos(x + y)$ .

### 1.3 – Functions, inverses and graphs of functions

1. The functions  $f(x)$  and  $g(x)$  are defined on suitable domains with

$$f(x) = \frac{3x-4}{x} \quad \text{and} \quad g(x) = \frac{4}{3-x}$$

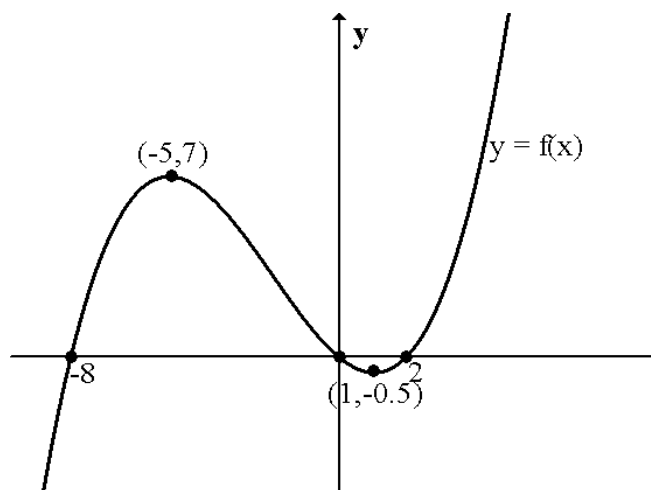
- (a) Find a formula for  $g(f(x))$ .  
(b) State the connection between  $f(x)$  and  $g(x)$ .

2.  $f(x) = x^2 - x - 12$  and  $g(x) = 3x + 1$

- (a) Find a formula for  $f(g(x))$ .  
(b) Solve  $f(g(x)) = 0$ .  
(c) State a suitable domain for the function  $h(x)$  where  $h(x) = \frac{1}{f(g(x))}$ .

3. The diagram opposite shows the graph of  $y = f(x)$ .

- (a) Sketch the graph of  $y = -f(x) + 3$   
(b) Sketch the graph of  $y = -3f(x - 2)$



### 1.4 – Vectors

1.  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{v} = 3\mathbf{i} + \mathbf{k}$ .

- (a) Find the vector  $3\mathbf{u} + \mathbf{v}$   
(b) Find the magnitude of vector  $2\mathbf{v} - \mathbf{u}$ .  
(c) Find a unit vector parallel to the vector  $2\mathbf{v} - \mathbf{u}$ .

2. A is the point  $(-1, 2, 0)$ , B is  $(3, 0, 6)$  and C is  $(9, -3, 15)$ .

Show that A, B and C are collinear stating the ratio of AB:BC.

3. The points  $P(0, 5, 9)$ ,  $Q(2, 3, 4)$  and  $R(6, u, v)$  are collinear. Find the values of  $u$  and  $v$ .

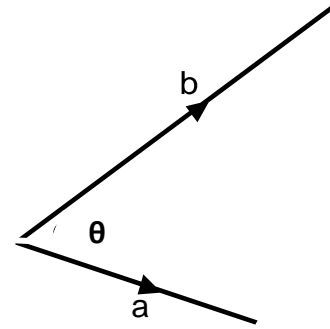
4. The points P, Q, R and S are  $(3, 1, -2)$ ,  $(-2, -4, 8)$ ,  $(0, -2, 4)$  and  $(4, 2, -4)$  respectively.

- (a) T divides PQ in the ratio 2:3. Find the coordinates of T.  
(b) Show that R, T and S are collinear.

5.  $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$  and  $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ . Show that the vectors  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular.

6. A triangle ABC has vertices A(2,1,-6), B(4,0,-1) and C(-5,2,3).  
Show that triangle ABC is right-angled at B.

7. Calculate the angle between the vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$



8. P is the point (4,0,-3), Q is (6,1,-1) and R is (14,0,-5).

Calculate the size of angle PQR.

9. In the diagram opposite, BCDE is a parallelogram and  $AE = ED$ .

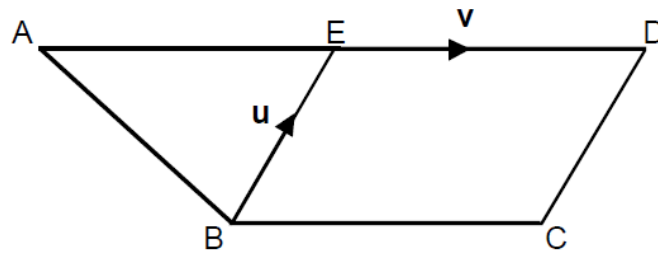
$$\overrightarrow{BE} = \mathbf{u} \text{ and } \overrightarrow{ED} = \mathbf{v}$$

Find, in terms of  $\mathbf{u}$  and  $\mathbf{v}$

(i)  $\overrightarrow{BD}$       (ii)  $\overrightarrow{AC}$

(iii)  $\overrightarrow{AF}$  where F divides AC in the ratio 2:1

(iv)  $\overrightarrow{FD}$



## Relationships and Calculus

### 1.1 – Polynomials and Quadratics

- (a) Express  $x^2 - 8x + 1$  in the form  $(x + a)^2 + b$ .

(b) Sketch the graph of  $y = x^2 - 8x + 1$ , showing clearly its turning point.
- (a) Express  $f(x) = 3x^2 + 12x - 2$  in the form  $f(x) = a(x + b) + c$ .

(b) Hence, or otherwise, write down the turning point of  $f(x)$  stating whether this turning point is a maximum or minimum.
- State the nature of the roots of:

(a)  $3x^2 - 2x - 5 = 0$                       (b)  $x^2 + 3x + 7 = 0$
- The roots of the equation  $(x + 1)(x + k) = -4$  are equal. Find the values of k.



5. The roots of the equation  $x^2 + kx - 3k = 4x - 7$  are real. Find the range of values for  $k$ .

6. Show that  $y = 2x^3 + x^2 + 9x + 1$  has no stationary points.

## **1.2 – Trigonometric Equations**

See 1.2 Expressions & Formulae for all trig. work – page 6

## **1.3 – Differentiation**

1. If  $f(x) = \frac{x^2-1}{\sqrt{x}}$  find  $f'(4)$ .

2.  $S = 3u(u^2 + 1)$ . Find the rate of change of  $S$  when  $u = \frac{4}{3}$

3. Find the equation of the tangent to the curve  $y = \frac{x^2(x^2-2)}{x}$  at the point where  $x = 2$ .

4. A tangent to the curve  $y = x^4 - 2x$  has a gradient of  $-6$ . Find the equation of this tangent.

5. Show that the curve  $y = x^3 - 6x^2 + 12x + 3$  is never decreasing.

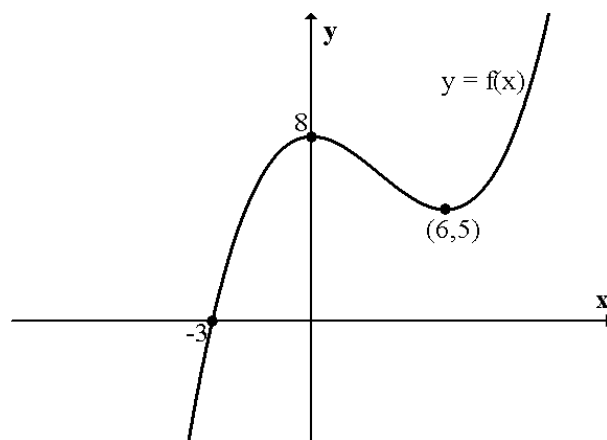
6. Find the values of  $x$  for which the curve  $f(x) = 2x^3 - 6x^2 - 48x + 5$  is strictly increasing.

7.  $f(x) = x^4 - 4x^3 + 5$ . Find the stationary points of  $f(x)$  and determine their nature.

8. Find the maximum and minimum values of  $f(x) = 2x^3 - 3x^2 - 12x$  in the range  $-3 \leq x \leq 3$ .

9. Shown opposite is the graph of  $y = f(x)$ .

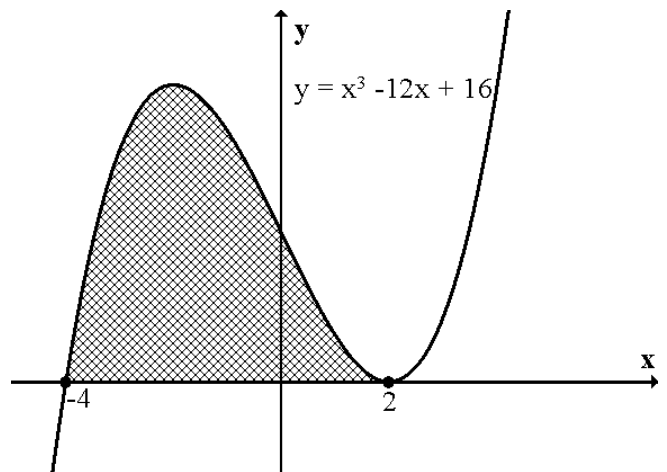
Sketch the graph of  $y = f'(x)$ .



## 1. 4 Integration

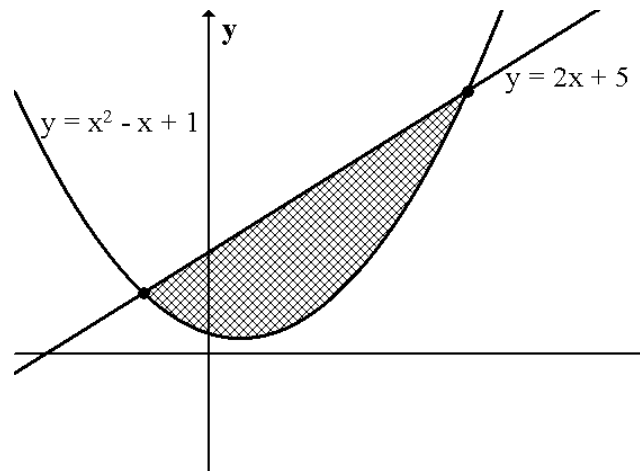
1.  $\frac{dy}{dx} = 3x^2 - 4x + 1$ . Find a formula for  $y$  given  $y = 2$  when  $x = -1$ .

2. Calculate the shaded area in the diagram shown opposite.



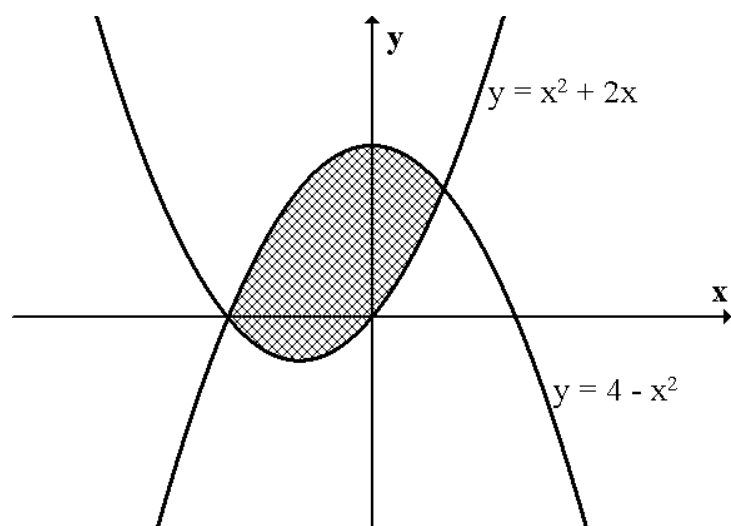
3. The diagram shows the line  $y = 2x + 5$  and the curve  $y = x^2 - x + 1$ .

Calculate the shaded area.



4. The diagram shows the parabolas  $y = x^2 + 2x$  and  $y = 4 - x^2$ .

Calculate the area enclosed by these two parabolas.

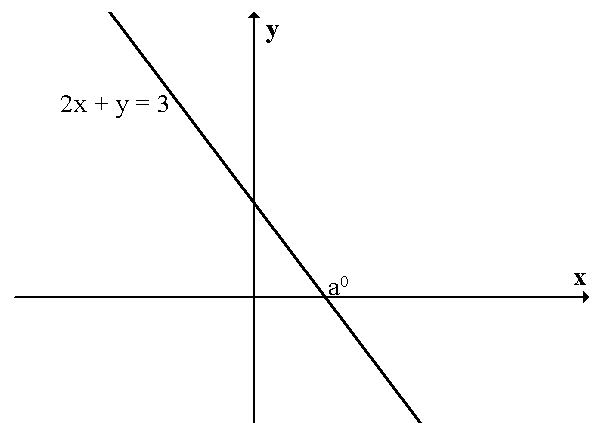


5. Find (a)  $\int \frac{x^3 - 1}{x^2} dx$  (b)  $\int_1^4 \sqrt{x}(\sqrt{x} - x) dx$

## Applications

### 1.1 – The Straight line

- Find the equation of the line parallel to the line  $3x + 2y - 10 = 0$  which passes through the point  $(-1, 4)$ .
- Find the equation of the line through the point  $(2, -5)$  perpendicular to the line AB where A is  $(4, 1)$  and B is  $(6, -3)$ .
- Find,  $a^\circ$ , the angle the line  $2x + y = 3$  makes with the positive direction of the  $x$ -axis.



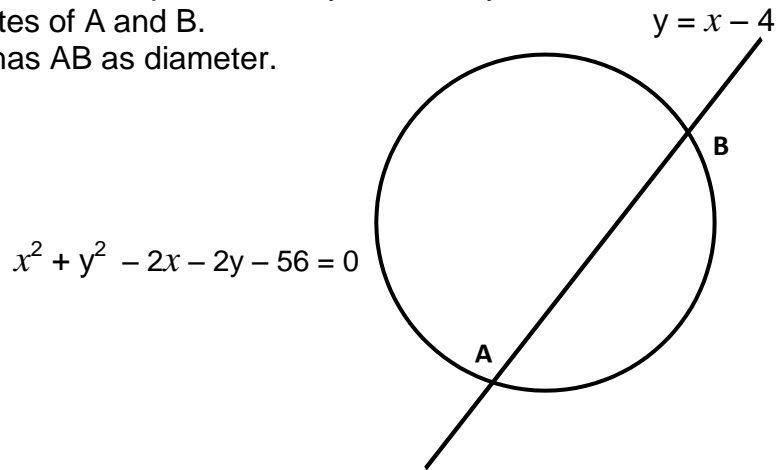
- Triangle ABC has vertices  $A(2, -5)$ ,  $B(8, 1)$  and  $C(7, 2)$ . Find the equation of the median from C.
- A is the point  $(2, -1)$ , B is  $(10, -5)$  and C is  $(6, 2)$ .
  - Find the equation of the perpendicular bisector of AB.
  - Find the equation of the altitude from B to AC.
  - Find the point of intersection of these lines.

### 1.2 – The Circle

- A circle has equation  $x^2 + y^2 - 6x + 2y - 35 = 0$ . Find the equation of the tangent to this circle at the point  $(-3, 2)$ .
- Find the equation of the circle which has PQ as diameter where P is  $(-2, 2)$  and Q is  $(6, 10)$ .

3. (a) The line  $y = x - 4$  intersects the circle with equation  $x^2 + y^2 - 2x - 2y - 56 = 0$  at two points A and B. Find the coordinates of A and B.

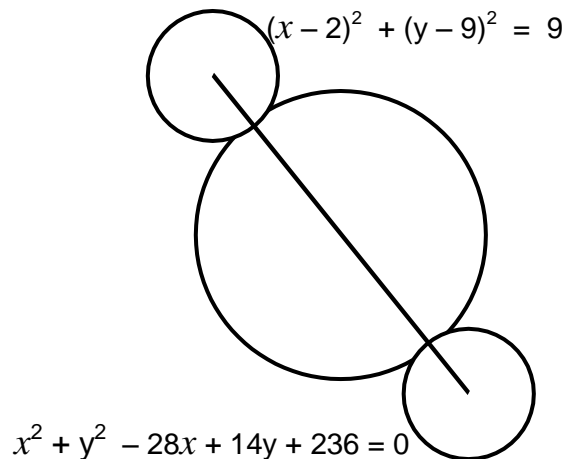
(b) Find the equation of the circle which has AB as diameter.



4. Prove that the line  $y = 2x + 6$  is a tangent to the circle with equation  $x^2 + y^2 - 8x + 2y - 28 = 0$  and find the point of contact.

5. The line  $y = x - 2$  intersects the circle  $x^2 + y^2 - 4x + 2y - 20 = 0$  at the points S and T. Find the coordinates of S and T.

6. Three circles touch externally as shown. The centres of the circles are collinear and the equations of the smaller circles are  $(x - 2)^2 + (y - 9)^2 = 9$  and  $x^2 + y^2 - 28x + 14y + 236 = 0$ . Find the equation of the larger circle.



### 1.3 – Recurrence Relations

1.  $u_{n+1} = 0.6u_n + 20$        $u_0 = 40$

(a) Find  $n$  such that  $u_n > 49$

(b) Explain why  $u_{n+1}$  has a limit and find the exact value of this limit.

2. A recurrence relation is defined as  $u_n = au_{n-1} + b$ .

The first three terms of this relation are 160, 200 and 230. Find the values of  $a$  and  $b$ .

3. A recurrence relation is  $u_{n+1} = 0.5u_n + 10$ . Given  $u_3 = 30$ , find the value of  $u_1$ .

4. Two sequences are defined by the recurrence relations

$$u_{n+1} = 0.4u_n + p \quad v_{n+1} = 0.6v_n + q$$

If both sequences have the same limit, express  $p$  in terms of  $q$ .

5. A patient is injected with 60 ml of an antibiotic drug. Every 4 hours 30% of the drug passes out of her bloodstream. To compensate for this an extra 20ml of antibiotic is given every 4 hours.

- (a) Find a recurrence relation for the amount of drug in the patient's bloodstream.
- (b) Calculate the amount of antibiotic remaining in the bloodstream after one day.

### **1.4 – Further Calculus**

1. Differentiate

(a)  $f(x) = (x^2 - 5)^4$

(b)  $y = \frac{2}{\sqrt{8x-3}}$

(c)  $f(x) = 2\sin(4x) - 2\cos^3x$

2.  $f(x) = 2\sin^2x$ . Find the value of  $f'(\frac{\pi}{4})$

3. A curve has equation  $y = \sqrt[3]{3x-1}$ . Find the equation of the tangent to this curve at the point where  $x = 3$ .

4. Find the equation of the tangent to the curve  $y = 4\sin(2x - \frac{\pi}{6})$  at the point where  $x = \frac{\pi}{2}$

5. Integrate

(a)  $\int (4x - 6)^3 dx$

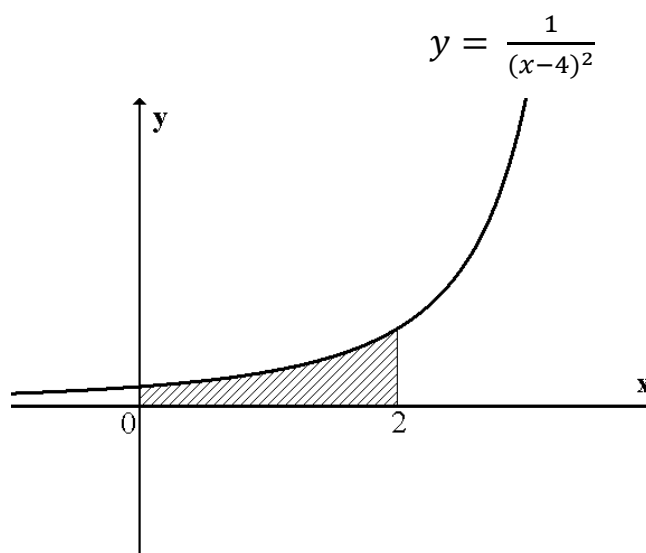
(b)  $\int 10\sqrt{1-6x} dx$

(c)  $\int 6\cos(2x - 3) dx$

6. Evaluate  $\int_1^3 \frac{8}{(2x-4)^2} dx$

7.  $\frac{dy}{dx} = 2\cos(4x)$ . This curve passes through the point  $(\frac{5}{12}\pi, \sqrt{3})$ . Find a formula for  $y$ .

8. The diagram shows part of the graph of  $y = \frac{1}{(x-4)^2}$ . Calculate the shaded area.



9. The diagram shows part of the graphs of  $y = \sin 2x$  and  $y = \cos x$ .
- Find the  $x$ -coordinates of A and B.
  - Calculate the shaded area.

